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# *THE DENSITY OF THE EARTH.*

BY PROFESSOR ASAPH HALL.

IN a paper recently published by the U. S. Coast and Geodetic Survey, Mr. C. A. Schott gives the results of a comparison of the geodetic and astronomical positions of sixty places on our eastern coast; these places extending from Calais, Maine, to Atlanta, Georgia. This comparison brings to light some curious differences in latitude and longitude; and if we consider the accuracy with which this survey is conducted, it is probable that the most of these differences are real, and are caused by local attractions and variations in the earth's crust. Thus at Washington, the plumb line is drawn $6''$ towards the west. Mr. Schott's interesting paper has recalled to mind an idea that occurred to me some time ago of determining the density of the earth by means of the tides in the Bay of Fundy. No doubt the same idea has occurred to others, and by a reference of Professor Wright, of Yale College, I find in Thomson and Tait's Natural Philosophy, Art. 818, an approximate formula for the effect of such a mass of water on the plumb line. The advantage of this method lies in the fact that the attracting mass is homogeneous and of known density, while in the case of a mountain, the density of the attracting mass is uncertain. On the other hand, the attraction of a mountain will generally be greater than can be got from the shifting of the tides, and the difficulty of measuring small angular quantities with the necessary accuracy is a serious hindrance in the application of the tidal method. This difficulty may be overcome perhaps by the employment of more refined apparatus, such as the horizontal pendulum. Another difficulty would be the determination of the form of the water, but probably this could be found from surveys and soundings. I confess, however, that it seems to me doubtful if any of these methods can ever give results as accurate as those found by Baily from his experiments made in such a complete manner with the Coulomb torsion balance. Still, the results found by independent methods have a value, and these methods ought not to be neglected.

In order to see how great would be the effect of a rectangular mass of water on a plumb line, let us take the axis of $x$ so that it bisects the upper surface of the water, and is at right angles to the edge. Let the origin be in the axis of $x$ at a distance $\delta$ from the edge; let $k$ be a constant depending on the unit and density of mass, and denote by $A$ the component of attraction in the axis of $x$. Then if $a$ is the distance from the origin to the farther side of the water, $2\beta$ the width of the rectangular mass, and $\gamma$ its depth, we shall have

$$A = k.\int_{\delta}^{a}\int_{-\beta}^{+\beta}\int_{0}^{\gamma}\frac{x.dx\,dy\,dz}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$$

This is an exact differential with respect to $x$, and after two integrations we have,

$$A = -k.\int_0^{\gamma} dz.\log[\beta+\surd(a^2+\beta^2+z^2)] + k.\int_0^{\gamma} dz.\log[-\beta+\surd(a^2+\beta^2+z^2)]$$
$$+k.\int_0^{\gamma} dz.\log[\beta+\surd(\delta^2+\beta^2+z^2)] - k.\int_0^{\gamma} dz.\log[-\beta+\surd(\delta^2+\beta^2+z^2)]. \quad \ldots (1)$$

These integrals are all of the same form, and an integration by parts gives,

$$\int dz.\log[\beta+\surd(n+z^2)] = z.\log[\beta+\surd(n+z^2)] - \int\frac{z^2.dz}{\beta\surd(n+z^2)+n+z^2}.$$

Adding $ndz - ndz$ to the numerator of the last term, and putting for the moment $n+z^2 = x^2$, we have the two forms

$$\int\frac{n.dx}{(x^2-n)^{\frac{1}{2}}.(\beta+x)}, \quad \text{and} \quad \int\frac{x^2.dx}{(x^2-n)^{\frac{1}{2}}.(\beta+x)}.$$

The first of these reduces to a known integral if we put $\beta + x = 1 \div y$; and if we add to the numerator of the second $\beta^2dx - \beta^2dx$ we have,

$$\int\frac{x^2.dx}{(x^2-n)^{\frac{1}{2}}(\beta+x)} = \int\frac{xdx}{(x^2-n)^{\frac{1}{2}}} - \beta.\int\frac{dx}{(x^2-n)^{\frac{1}{2}}} + \beta^2.\int\frac{dx}{(x^2-n)^{\frac{1}{2}}(\beta+x)}.$$

These integrals are also known, and we have

$$\int dz.\log[\beta+\surd(a^2+\beta^2+z^2)] = z.\log[\beta+\surd(a^2+\beta^2+z^2)]$$
$$+a.\text{arc tan.}\frac{-a^2-\beta^2-\beta\surd(a^2+\beta^2+z^2)}{az} - z + \beta.\log[z+\surd(a^2+\beta^2+z^2)]$$
$$+ \text{ a constant.}$$

In the case therefore of a homogeneous rectangular mass, the attraction can be found exactly; and by uniting terms having the same factor we have,

$$A = 2k\beta.\log\frac{[\gamma+\surd(\delta^2+\beta^2+\gamma^2)].\surd(\alpha^2+\beta^2)}{[\gamma+\surd(\alpha^2+\beta^2+\gamma^2)].\surd(\delta^2+\beta^2)}$$

$$+k\gamma.\log\frac{[\beta+\surd(\delta^2+\beta^2+\gamma^2)].[-\beta+\surd(\alpha^2+\beta^2+\gamma^2)]}{[\beta+\surd(\alpha^2+\beta^2+\gamma^2)].[-\beta+\surd(\delta^2+\beta^2+\gamma^2)]}$$

$$+k\alpha.\text{arc tan}\,\frac{2\alpha\,\beta\,\gamma\surd(\alpha^2+\beta^2+\gamma^2)}{\alpha^2(\alpha^2+\beta^2+\gamma^2)-\beta^2\gamma^2} - k\delta.\text{arc tan}\,\frac{2\delta\,\beta\,\gamma\surd(\delta^2+\beta^2+\gamma^2)}{\delta^2(\delta^2+\beta^2+\gamma^2)-\beta^2\gamma^2}.$$

If now we rationalize the second logarithmic term, put

$$s = \surd(\alpha^2 + \beta^2 + \delta^2), \qquad s_1 = \surd(\delta^2 + \beta^2 + \gamma^2),$$

and change the arcs to half arcs, we have finally,

$$A = 2k\beta.\log\frac{(\gamma+s_1).(\alpha^2+\beta^2)^{\frac{1}{2}}}{(\gamma+s\ ).(\delta^2+\beta^2)^{\frac{1}{2}}} + 2k\gamma.\log\frac{(\beta+s_1).(\alpha^2+\gamma^2)^{\frac{1}{2}}}{(\beta+s\ ).(\delta^2+\gamma^2)^{\frac{1}{2}}}$$

$$+2k\alpha.\text{arc tan}\,\frac{\beta\gamma}{\alpha s} - 2k\delta.\text{arc tan}\,\frac{\beta\gamma}{\delta s}. \tag{2}$$

Since $z$ is a small quantity in nearly all practical cases, if we neglect $z^2$ in equation (1) and then integrate we have

$$A = k\gamma.\log\frac{[\beta+\surd(\delta^2+\beta^2)].[-\beta+\surd(\alpha^2+\beta^2)]}{[\beta+\surd(\alpha^2+\beta^2)].[-\beta+\surd(\delta^2+\beta^2)]}. \tag{3}$$

This expression is similar to that given by Thomson and Tait, and also by Pratt in his book on the Figure of the Earth. This formula gives the principal part of the attraction, and it may be written

$$A = 2k\gamma.\log\frac{\alpha\beta + \alpha\surd(\delta^2 + \beta^2)}{\delta\beta + \delta\surd(\alpha^2 + \beta^2)}.$$

The logarithms in the preceding expressions are Napierian; and if we use the common tables we must multiply their logarithms by 2.302585.

In order to apply these formulæ to a case that will be nearly represented by the tides in the Bay of Fundy, I take one mile for the unit of length, and put

$$\alpha = 30; \ \beta = 15; \ \gamma = \delta = 0.01 = 53 \text{ feet nearly.}$$

The terms of formula (2) give

first term = + 0.011122.$k$

second " = + 0.143571 "

third " = + 0.008944 "

fourth " = — 0.008411 "

sum = + 0.155226.$k$

If we denote the force of gravity of the earth by $g$, we have by Clairaut's formula,

$$g = G.[1 + (\tfrac{5}{2}m - \varepsilon)\sin\varphi^2].$$

$G$ is the force of gravity at the Equator; $m$ is the ratio of the centrifugal force at the Equator to gravity; $\varepsilon$ is the flattening of the Earth, and $\varphi$ is the latitude of the place. Denoting by $k'$ the constant of the Earth's mass, depending on the unit of mass and its density, we have,

$$G = \frac{4\pi}{3}k'.(a^2b)^{\frac{1}{3}}: \qquad m = \frac{1}{289}.$$

$$\varepsilon = \frac{a - b}{a} = \frac{13.5}{3963.3},$$

and the coefficient of sine $\varphi^2$ is 0.005244. If we put $\varphi = 45°$,

$$g = 16626.04.k'.$$

The mean density of the earth according to Baily is 5.6747; and that of sea water being 1.026,

$$k' = \frac{5.6747}{1.026}.k = 5.5309 . k.$$

If $v$ be the variation of the plumb line produced by the tide, then

$$v = \frac{0.155226}{16626.04 \times 5.5309} = 0''.348.$$

It would be difficult to determine this small arc with the necessary accuracy.

Baily's value of the mean density was found from a long and very careful series of experiments made in London in 1841 and 1842. The two attracting masses were balls of lead 12 inches in diameter, and weighing 389 pounds each. The small balls placed at the ends of the torsion rods were 2 inches in diameter, and made of different substances, platinum, lead, zinc, glass, ivory and brass. From all his experiments Baily found

$$\text{mean density} = 5.6747 \pm 0.0038;$$

the unit of density being that of distilled water. The computed probable error of this result is very small, being only $\frac{7}{10000}$ of the whole value.

It would hardly be possible to determine this quantity from the attraction of a mountain or mass of water to such a degree of accuracy. On the other hand, it must be remembered that the theory of probable errors does not recognize the existence of constant errors in the experiments, and that conclusions drawn from the values of probable errors are frequently erroneous. It would be interesting to have this determination repeated under different circumstances.